J. Myanmar Acad. Arts Sci. 2020 **Vol. XVIII.No.2**B

LIE DERIVATIVE, KILLING EQUATION AND KILLING VECTOR FIELDS IN SPACETIMES STRUCTURE

Min Thaw Tar[1], Naing Naing Wint Htoon[2], Yee May Thwin[3], Thant Zin Naing[4]

Abstract

Attempts have been made to explore the physical properties of black hole mechanics such as Killing equation and Killing vector fields and its applications to Minkowski spacetime, static spacetimes and spherically symmetric spacetimes. As the situation dictates, Mathematica software is used to utilized for detailed computations and visualization of the results.

Keywords: Killing equation, Killing vector fields, Static spherically symmetric spacetimes.

Introduction

The Lie derivative evaluates the change of a tensor field, along the flow defined by another vector field. This change is coordinate invariant and therefore the Lie derivative is defined on any differentiable manifold. Functions, tensor fields and forms can be differentiated with respect to a vector field.

Killing fields are the infinitesimal generators of isometries; that is, flows generated by Killing fields are continuous isometries of the manifold. More simply, the flow generates a symmetry, in the sense that moving each point on an object the same distance in the direction of the Killing vector field will not distort distances on the object.

The Lie Derivative

With or without the covariant derivative, which requires a connection on all of spacetime, there is another sort of derivation called the Lie derivative, which requires only a curve.

Let $C : R \to M$ be a curve in M with tangent vectors, $\xi = \dfrac{d}{d\lambda}$, with components

$$\xi = \xi^{\mu} \frac{\partial}{\partial x^{\mu}} = \frac{dx^{\mu}}{\partial \lambda} \frac{\partial}{\partial x^{\mu}} \tag{1}$$

The Lie derivative generalizes the directional derivative of a function,

$$\frac{df}{d\lambda} = \xi^{\mu} \frac{\partial f}{\partial x^{\mu}} \tag{2}$$

to higher rank tensors. First, consider a vector field, v, defined on M. One defines the Lie derivative of v at a point P along C to be

$$L_{\xi} v = \lim_{\varepsilon \to 0} \frac{v(P+\varepsilon\xi) - v(P)}{\varepsilon} \tag{3}$$

where $v(P + \varepsilon\xi)$ is the Lie transport of v along the curve. For simplicity, let $P = C$ ($\lambda = 0$). Lie transport involves taking the value of the vector field at a point on C, say, v (λ), and performing a

[1] PhD Candidate, Department of Physics , University of Yangon.
[2] Lecturer, Department of Physics, University of Yangon.
[3] Lecturer, Department of Physics, Magway University.
[4] Retired Pro-Rector (Admin), International Theravāda Buddhist Missionary University, Yangon.

coordinate transformation to bring the point C (λ) back to P = C (0) (Adler.R, Bazin.M, Schiffer.M, 1975). The coordinate transformation one require is, for infinitesimal $\lambda = \varepsilon$,

$$y^{\alpha} = x^{\alpha} - \varepsilon\xi^{\alpha}(0) \tag{4}$$

The components of v^{α} change as

$$\tilde{v}^{\alpha}(0) = v^{\beta}(\lambda)\frac{\partial y^{\alpha}}{\partial x^{\beta}} \tag{5}$$

$$= \left[v^{\beta}(x^{\mu}(0) + \varepsilon\xi^{\mu})\right]\ (\delta^{\alpha}_{\beta} - \varepsilon\partial_{\beta}\xi^{\alpha})$$

$$= \left[v^{\beta}(0) + \varepsilon\xi^{\mu}\partial_{\mu}v^{\beta}(0)\right]\ (\delta^{\alpha}_{\beta} - \varepsilon\partial_{\beta}\xi^{\alpha})$$

$$= v^{\alpha}(0) + \varepsilon\xi^{\mu}\partial_{\mu}v^{\alpha}(0) - \varepsilon\ \ v^{\beta}(0)\partial_{\beta}\xi^{\alpha} \tag{6}$$

The derivative is then

$$L_{\xi}v = \lim_{\varepsilon\to 0}\frac{v(P+\varepsilon\xi) - v(P)}{\varepsilon} \tag{7}$$

$$L_{\xi}v = \lim_{\varepsilon\to 0}\frac{v^{\alpha}(0) + \varepsilon\xi^{\mu}\partial_{\mu}v^{\alpha}(0) - \varepsilon v^{\beta}(0)\partial_{\beta}\xi^{\alpha} - v(0)}{\varepsilon}$$

$$L_{\xi}v = \xi^{\mu}\partial_{\mu}v^{\alpha}(0) - v^{\beta}(0)\partial_{\beta}\xi^{\alpha} \tag{8}$$

An easy proof of the covariance of this result is that it equals the commutator of the two vectors,

$$L_{\xi}v = [\xi, v] \tag{9}$$

which has the same form when ξ and v are expanded in components,

$$[\xi, v] = [\xi^{\alpha}\partial_{\alpha}, v^{\beta}\partial_{\beta}] \tag{10}$$

$$= \xi^{\alpha}\partial_{\alpha}v^{\beta}\partial_{\beta} - v^{\beta}\partial_{\beta}\xi^{\alpha}\partial_{\alpha}$$

$$= (\xi^{\beta}\partial_{\beta}v^{\alpha} - v^{\beta}\partial_{\beta}\xi^{\alpha})\partial_{\alpha} \tag{11}$$

The generalization to higher rank tensors is immediate because derivations must satisfy the Leibnitz rule (Hawking S.W, Ellis G.F.R, 1973) .Thus, for an outer product of two vectors,

$$T^{\alpha\beta} = u^{\alpha}v^{\beta} \tag{12}$$

one has

$$L_\xi T^{\alpha\beta} = L_\xi (u^\alpha v^\beta) \quad (13)$$

$$L_\xi T^{\alpha\beta} = (L_\xi u^\alpha) v^\beta + u^\alpha (L_\xi v^\beta)$$

$$L_\xi T^{\alpha\beta} = (\xi^\mu \partial_\mu u^\alpha - u^\mu \partial_\mu \xi^\alpha) v^\beta + u^\alpha (\xi^\mu \partial_\mu v^\beta - v^\mu \partial_\mu \xi^\beta)$$

$$L_\xi T^{\alpha\beta} = \xi^\mu \partial_\mu (u^\alpha v^\beta) - u^\mu v^\beta \partial_\mu \xi^\alpha - u^\alpha v^\mu \partial_\mu \xi^\beta$$

$$L_\xi T^{\alpha\beta} = \xi^\mu \partial_\mu T^{\alpha\beta} - T^{\alpha\beta} \partial_\mu \xi^\alpha - T^{\alpha\beta} \partial_\mu \xi^\beta \quad (14)$$

and so on for higher ranks, with one correction term, $-T^{\alpha\ldots\mu\ldots\beta}\partial_\mu \xi^\nu$, for each index .

For forms, one uses the directional derivative of a scalar,

$$L_\xi \phi = \xi^\mu \frac{\partial \phi}{\partial x^\mu} \quad (15)$$

together with $\phi = v^\alpha w_\alpha$, for arbitrary v^α,

$$\xi^\mu \frac{\partial (v^\alpha w_\alpha)}{\partial x^\mu} = L_\xi (v^\alpha w_\alpha) \quad (16)$$

$$\xi^\mu (\partial_\mu v^\alpha) w_\alpha + v^\alpha \xi^\mu \partial_\mu w_\alpha = (L_\xi v^\alpha) w_\alpha + v^\alpha L_\xi w_\alpha$$

$$\xi^\mu (\partial_\mu v^\alpha) w_\alpha + v^\alpha \xi^\mu \partial_\mu w_\alpha = \xi^\beta (\partial_\beta v^\alpha) w_\alpha - v^\beta (\partial_\beta \xi^\beta) w_\alpha + v^\alpha L_\xi w_\alpha$$

$$v^\alpha \xi^\mu \partial_\mu w_\alpha = -v^\beta (\partial_\beta \xi^\beta) w_\alpha + v^\alpha L_\xi w_\alpha$$

$$v^\alpha L_\xi w_\alpha = v^\alpha \xi^\mu \partial_\mu w_\alpha + v^\alpha (\partial_\alpha \xi^\beta) w_\beta \quad (17)$$

Since this must hold for all v^α ,

$$L_\xi w_\alpha = \xi^\mu \partial_\mu w_\alpha + w_\beta \partial_\alpha \xi^\beta \quad (18)$$

Symmetries of Minkowski spacetime

Consider flat spacetime, for which the metric is Minkowski, $\eta_{\mu\nu}$. In Cartesian coordinates,

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad (19)$$

and the Christoffel connection vanishes, $\Gamma^\alpha{}_{\mu\nu} = 0$. Then one may replace the covariant derivatives by partial derivatives, and the Killing equation is simply

$$\xi_{\alpha,\beta} + \xi_{\beta,\alpha} = 0 \quad (20)$$

Taking a further derivative, one has

$$\xi_{\alpha,\beta\mu} + \xi_{\beta,\alpha\mu} = 0 \tag{21}$$

Now, cycle the indices twice, to give

$$\xi_{\alpha,\beta\mu} + \xi_{\beta,\alpha\mu} = 0 \tag{22}$$

$$\xi_{\beta,\mu\alpha} + \xi_{\mu,\beta\alpha} = 0 \tag{23}$$

$$\xi_{\mu,\alpha\beta} + \xi_{\alpha,\mu\beta} = 0 \tag{24}$$

Adding equation (31) and (32) and subtracting equation (33) one finds

$$0 = \xi_{\alpha,\beta\mu} + \xi_{\beta,\alpha\mu} + \xi_{\beta,\mu\alpha} + \xi_{\mu,\beta\alpha} - \xi_{\mu,\alpha\beta} - \xi_{\alpha,\mu\beta}$$

$$0 = \xi_{\beta,\alpha\mu} \tag{25}$$

so that the second derivative of ξ_β vanishes. This means that ξ_β must be linear in the coordinates,

$$\xi_\alpha = a_\alpha + b_{\alpha\beta} x^\beta \tag{26}$$

Substituting this into the Killing equation,

$$0 = \xi_{\alpha,\beta} + \xi_{\beta,\alpha} \tag{27}$$

$$0 = b_{\alpha\beta} + b_{\beta\alpha} \tag{28}$$

so that a_α is arbitrary while $b_{\alpha\beta}$ must be antisymmetric. One therefore finds exactly 10 isometries in Minkowski space. This is the maximum number of independent solutions to the Killing equation. The static, spherically symmetric Schwarzschild solution had one timelike Killing field and three spatial rotational Killing fields for a total of three. A generic spacetime has no isometries (Schutz. B. F, 2009).

Static, Spherically Symmetric Spacetimes

One may now say what one means by a static, spherically symmetric spacetime. To be static, there must be a timelike Killing vector field; to be spherically symmetric, one require a full set of three rotational (hence spacelike) Killing vectors. one use the Lie derivative to say restrict the form of the metric for a static, spherically symmetric spacetime. If one want a statics spacetime, it means that one want there to exist a timelike Killing vector field. Choosing the time coordinate to be the parameter t = λ, the symmetry condition becomes

$$\begin{aligned} 0 &= L_\xi g_{\alpha\beta} \\ &= \xi^\mu \partial_\mu g_{\alpha\beta} + \partial_\alpha \xi^\mu g_{\mu\beta} + \partial_\beta \xi^\mu g_{\alpha\mu} \end{aligned} \tag{29}$$

However, with $x^0 = t = \lambda$, the components of ξ are constant, so that

$$\partial_\alpha \xi^\mu = 0 \tag{30}$$

Therefore,

$$0 = \xi^\mu \partial_\mu g_{\alpha\beta} \tag{31}$$

$$= \frac{\partial}{\partial t}(g_{\alpha\beta})$$

and we have a coordinates system in which the metric is independent of the time coordinate.

For the spherical symmetry, we know that we have three rotational Killing vector fields which together generate SO(3). We can pick two of these for coordinates, but they will not commute with one another, so the metric will not be independent of both coordinates. Starting with the familiar form

$$\xi_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$\xi_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

$$\xi_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

it is natural to choose one coordinate, φ, such that

$$\frac{\partial}{\partial \varphi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{32}$$

is a Killing vector. To describe a second direction, we want a linear combination of the remaining two rotations,

$$\alpha(\varphi)\xi_1 + \beta(\varphi)\xi_2 \tag{33}$$

and we want this to remain orthogonal to ξ_3,

$$0 = \langle \xi_3, \alpha\xi_1 + \beta\xi_2 \rangle \tag{34}$$

$$= \left\langle x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \alpha\left(y\frac{\partial}{\partial} - z\frac{\partial}{\partial y}\right) + \beta\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\rangle$$

$$= x\left\langle \frac{\partial}{\partial y}, \alpha\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \beta\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\rangle$$

$$- y\left\langle \frac{\partial}{\partial x}, \alpha\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \beta\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\rangle$$

$$= x\left\langle \frac{\partial}{\partial y}, -\alpha z\frac{\partial}{\partial y}\right\rangle - y\left\langle \frac{\partial}{\partial x}, \beta z\frac{\partial}{\partial x}\right\rangle$$

$$= \alpha z x\left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle - \beta z y\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle$$

$$= -z(\alpha x + \beta y)$$

$$= -r\, sin\theta z\, (\alpha \cos\varphi + \beta \sin\varphi)$$

To get zero, one can take

$$\alpha = sin\ \varphi$$

$$\beta = -\cos\varphi$$

Then we have

$$\xi_4 = \xi_1 \sin\varphi + \xi_2 \cos\varphi \tag{35}$$

$$= sin\ \varphi\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) - \cos\varphi\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$= sin\ \varphi\left(r\ sin\ \theta\ sin\ \varphi\frac{\partial}{\partial z} - r\cos\theta\frac{\partial}{\partial y}\right) - \cos\varphi\left(r\cos\theta\frac{\partial}{\partial x} - r\sin\theta\cos\varphi\frac{\partial}{\partial z}\right)$$

$$= r\sin\theta\sin\varphi\ \frac{\partial}{\partial z} - \mathrm{r}\cos\theta\sin\varphi\frac{\partial}{\partial y} - r\cos\varphi\cos\theta\ \frac{\partial}{\partial x} + r\sin\theta\cos\varphi\ \frac{\partial}{\partial z}$$

$$= -\cos\theta\left(r\cos\varphi\frac{\partial}{\partial x} + r\sin\varphi\frac{\partial}{\partial y}\right) + \mathrm{r}\sin\theta\frac{\partial}{\partial z}$$

$$= -\cos\theta\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - \mathrm{r}\sin\theta\ \frac{\partial}{\partial z}$$

Compare

$$\frac{\partial}{\partial x} = \frac{x}{r}\ \frac{\partial}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\ \frac{xz}{r^2}\ \frac{\partial}{\partial\theta} - \ \frac{y}{x^2+y^2}\ \frac{\partial}{\partial\varphi} \tag{36}$$

$$= \sin\theta\cos\varphi\ \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\ \frac{\partial}{\partial\theta} - \frac{1}{r}\frac{sin\varphi}{sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y} = \frac{y}{r}\frac{\partial}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{yz}{r^2}\frac{\partial}{\partial\theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial\varphi} \tag{37}$$

$$= \sin\theta\sin\varphi\ \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\ \frac{\partial}{\partial\theta} + \frac{1}{r}\frac{cos\varphi}{sin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial z} = \frac{z}{r}\frac{\partial}{\partial r} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial}{\partial\theta} \tag{38}$$

$$= cos\theta\frac{\partial}{\partial r} - \frac{sin\theta}{r}\frac{\partial}{\partial\theta}$$

so we have

$$\xi_4 = -cos\theta\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + r\ sin\theta\frac{\partial}{\partial x} \tag{39}$$

$$= cos\theta\left(r\sin\theta\ cos^2\varphi\frac{\partial}{\partial r} + cos\ \theta\ cos^2\varphi\frac{\partial}{\partial\theta} + r\ sin\theta\ sin^2\ \varphi\frac{\partial}{\partial\theta} + cos\theta sin^2\varphi\frac{\partial}{\partial\theta}\right)$$
$$- r\ sin\theta\left(cos\theta\frac{\partial}{\partial r} - \frac{sin\ \theta}{\partial r}\frac{\partial}{\partial\theta}\right)$$

$$= r\ sin\theta\cos\theta\frac{\partial}{\partial r} + cos^2\ \theta\frac{\partial}{\partial\theta} - r\sin\theta\cos\theta\ \frac{\partial}{\partial r} + sin^2\ \theta\ \frac{\partial}{\partial\theta}$$

$$= r\ sin\theta\ cos\theta \frac{\partial}{\partial r} + cos^2\theta \frac{\partial}{\partial \theta} - r\ sin\theta\ cos\theta\ \frac{\partial}{\partial r} + sin^2 \frac{\partial}{\partial \theta}$$

$$= cos^2\theta \frac{\partial}{\partial \theta} + sin^2\theta \frac{\partial}{\partial \theta}$$

$$= \frac{\partial}{\partial \theta}$$

We may therefore take two of the Killing vectors to be

$$\xi_4 = \frac{\partial}{\partial \theta} \tag{40}$$

$$\xi_3 = \frac{\partial}{\partial \varphi} \tag{41}$$

giving two coordinates, $\theta, \varphi,$ corresponding to symmetry directions. Since these do not commute, the metric cannot be independent of both.

Static, Spherically Symmetric Spacetimes

One may now say what one means by a static, spherically symmetric spacetime. To be static, there must be a timelike Killing vector field; to be spherically symmetric, one require a full set of three rotational (hence spacelike) Killing vectors. one use the Lie derivative to say restrict the form of the metric for a static, spherically symmetric spacetime. If one want a statics spacetime, it means that one want there to exist a timelike Killing vector field. Choosing the time coordinate to be the parameter t = λ, the symmetry condition becomes

$$0 = L_\xi g_{\alpha\beta} \tag{36}$$

$$= \xi^\mu \partial_\mu g_{\alpha\beta} + \partial_\alpha \xi^\mu g_{\mu\beta} + \partial_\beta \xi^\mu g_{\alpha\mu}$$

However, with $x^0 = t = \lambda$, the components of ξ are constant, so that

$$\partial_\alpha \xi^\mu = 0 \tag{37}$$

Therefore,

$$0 = \xi^\mu \partial_\mu g_{\alpha\beta} \tag{38}$$

$$= \frac{\partial}{\partial t}(g_{\alpha\beta})$$

and we have a coordinates system in which the metric is independent of the time coordinate.

For the spherical symmetry, we know that we have three rotational Killing vector fields which together generate SO(3). We can pick two of these for coordinates, but they will not commute with one another, so the metric will not be independent of both coordinates. Starting with the familiar form

$$\xi_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$\xi_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

$$\xi_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

it is natural to choose one coordinate, φ, such that

$$\frac{\partial}{\partial\varphi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{39}$$

is a Killing vector. To describe a second direction, we want a linear combination of the remaining two rotations,

$$\alpha(\varphi)\xi_1 + \beta(\varphi)\xi_2 \tag{40}$$

and we want this to remain orthogonal toξ_3,

$$\begin{aligned}
0 &= \langle \xi_3, \alpha\xi_1 + \beta\xi_2\rangle \qquad (41)\\
&= \left\langle x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \alpha\left(y\frac{\partial}{\partial} - z\frac{\partial}{\partial y}\right) + \beta\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\rangle\\
&= x\left\langle\frac{\partial}{\partial y}, \alpha\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \beta\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\rangle\\
&\quad - y\left\langle\frac{\partial}{\partial x}, \alpha\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \beta\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\rangle\\
&= x\left\langle\frac{\partial}{\partial y}, -\alpha z\frac{\partial}{\partial y}\right\rangle - y\left\langle\frac{\partial}{\partial x}, \beta z\frac{\partial}{\partial x}\right\rangle\\
&= \alpha z x\left\langle\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle - \beta z y\left\langle\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle\\
&= -z(\alpha x + \beta y)\\
&= -r\ sin\theta z\ (\alpha\cos\varphi + \beta\sin\varphi)
\end{aligned}$$

To get zero, one can take

$$\alpha = sin\ \ \varphi$$

$$\beta = -\cos\varphi$$

Then we have

$$\begin{aligned}
\xi_4 &= \xi_1\sin\varphi + \xi_2\cos\varphi \qquad (42)\\
&= sin\ \varphi\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) - \cos\varphi\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\\
&= sin\ \varphi\left(r\ sin\ \theta\ sin\ \varphi\frac{\partial}{\partial z} - r\cos\theta\frac{\partial}{\partial y}\right) - \cos\varphi\left(r\cos\theta\frac{\partial}{\partial x} - r\sin\theta\cos\varphi\frac{\partial}{\partial z}\right)\\
&= r\sin\theta\sin\varphi\ \frac{\partial}{\partial z} - \mathrm{r}\cos\theta\sin\varphi\frac{\partial}{\partial y} - r\cos\varphi\cos\theta\ \frac{\partial}{\partial x} + r\sin\theta\cos\varphi\ \frac{\partial}{\partial z}\\
&= -\cos\theta\left(r\cos\varphi\frac{\partial}{\partial x} + r\sin\varphi\frac{\partial}{\partial y}\right) + \mathrm{r}\sin\theta\frac{\partial}{\partial z}\\
&= -\cos\theta\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - \mathrm{r}\sin\theta\ \frac{\partial}{\partial z}
\end{aligned}$$

Compare

$$\frac{\partial}{\partial x} = \frac{x}{r}\frac{\partial}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{xz}{r^2}\frac{\partial}{\partial \theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \varphi} \tag{43}$$

$$= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{sin\varphi}{sin\theta}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = \frac{y}{r}\frac{\partial}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{yz}{r^2}\frac{\partial}{\partial \theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \varphi} \tag{44}$$

$$= \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{cos\varphi}{sin\theta}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z} = \frac{z}{r}\frac{\partial}{\partial r} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial}{\partial \theta} \tag{45}$$

$$= cos\theta\frac{\partial}{\partial r} - \frac{sin\theta}{r}\frac{\partial}{\partial \theta}$$

so we have

$$\xi_4 = -cos\theta\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + r\ sin\theta\frac{\partial}{\partial x} \tag{46}$$

$$= cos\theta\left(r\sin\theta\ cos^2\varphi\frac{\partial}{\partial r} + cos\ \theta\ cos^2\varphi\frac{\partial}{\partial \theta} + r\ sin\theta\ sin^2\ \varphi\frac{\partial}{\partial \theta} + cos\theta sin^2\varphi\frac{\partial}{\partial \theta}\right)$$
$$- r\ sin\theta\left(cos\theta\frac{\partial}{\partial r} - \frac{sin\ \theta}{\partial r}\frac{\partial}{\partial \theta}\right)$$

$$= r\ sin\theta\cos\theta\frac{\partial}{\partial r} + cos^2\ \theta\frac{\partial}{\partial \theta} - r\sin\theta\cos\theta\ \frac{\partial}{\partial r} + sin^2\ \theta\ \frac{\partial}{\partial \theta}$$

$$= r\ sin\theta\ cos\theta\frac{\partial}{\partial r} + cos^2\theta\frac{\partial}{\partial \theta} - r\ sin\theta\ cos\theta\ \frac{\partial}{\partial r} + sin^2\frac{\partial}{\partial \theta}$$

$$= cos^2\theta\frac{\partial}{\partial \theta} + sin^2\theta\frac{\partial}{\partial \theta}$$

$$= \frac{\partial}{\partial \theta}$$

We may therefore take two of the Killing vectors to be

$$\xi_4 = \frac{\partial}{\partial \theta} \tag{47}$$

$$\xi_3 = \frac{\partial}{\partial \varphi} \tag{48}$$

giving two coordinates, $\theta, \varphi,$ corresponding to symmetry directions. Since these do not commute, the metric cannot be independent of both.

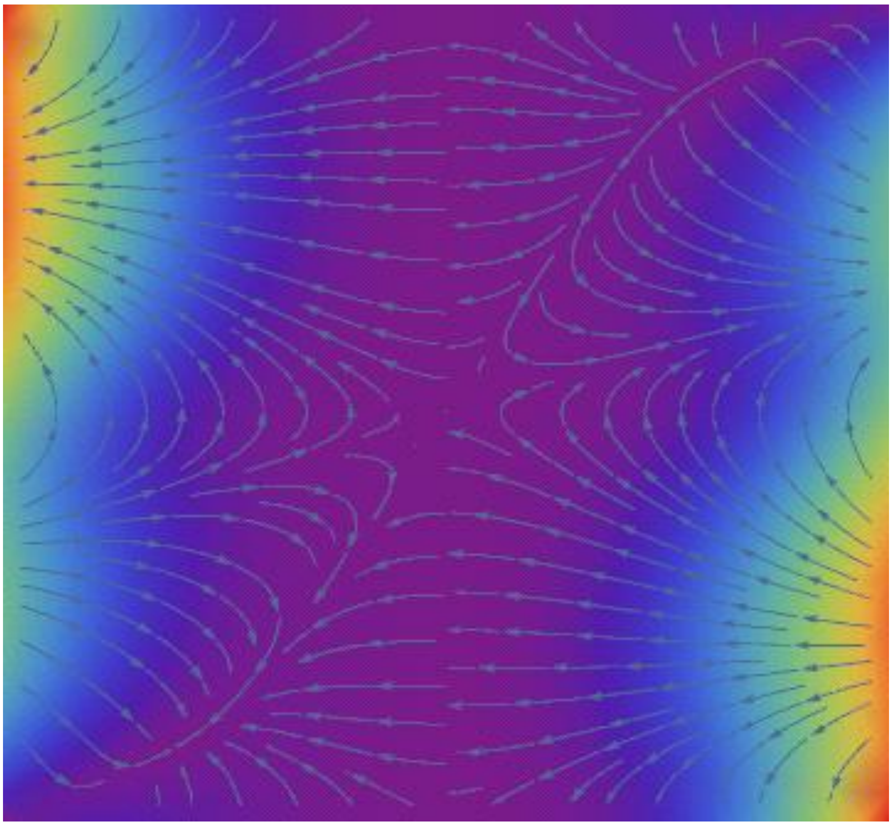

Figure 1 The typical visualization of the gradient of a vector field $\boldsymbol{\xi_1}$

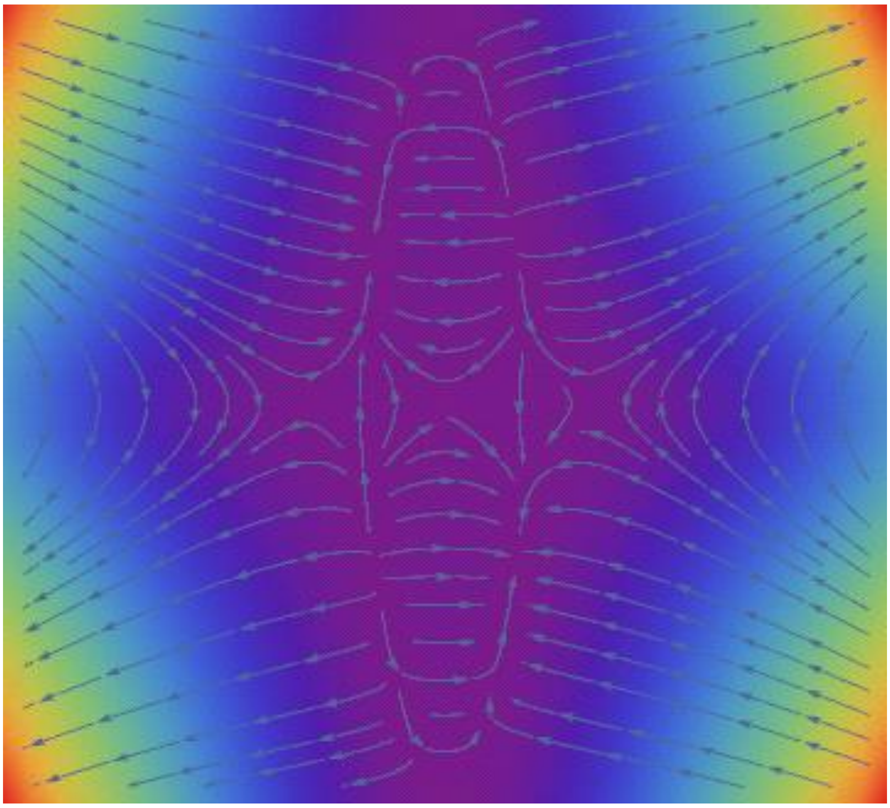

Figure 2 The typical visualization of the gradient of a vector field $\boldsymbol{\xi_2}$

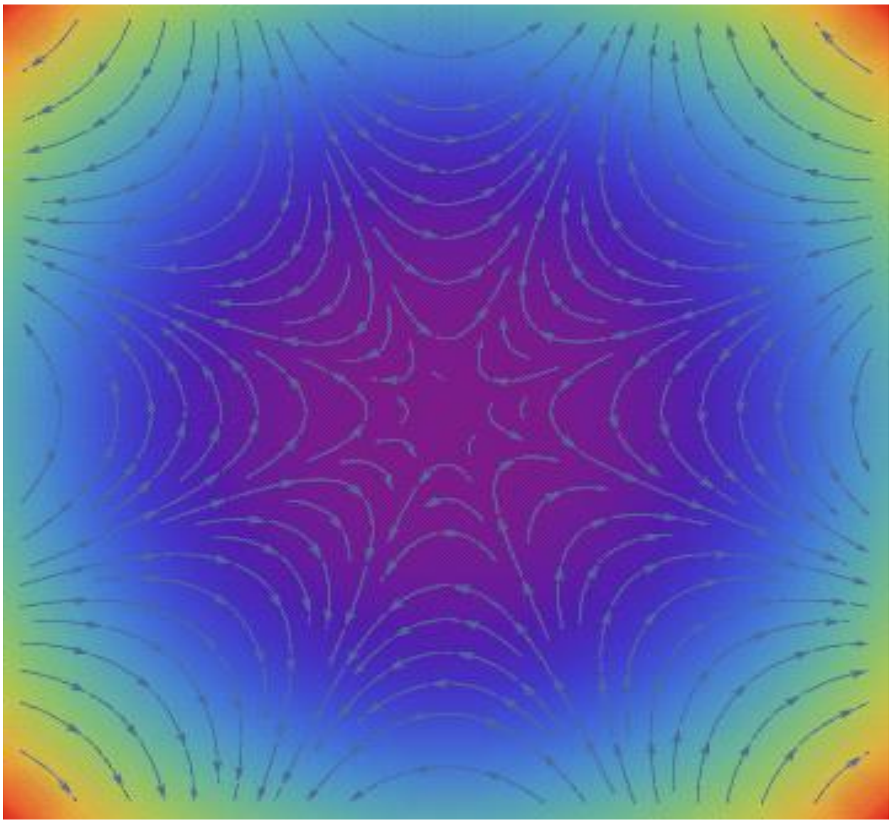

Figure 3 The typical visualization of the gradient of a vector field ξ_3

Concluding Remarks

In this paper, attempts have been made to explore the interesting physical properties of spacetimes structure. Lie derivative , Killing field equations and utilizations of these are also been presented. The symmetry of Minkowki spacetimes, static spacetimes and spherically symmetric spacetimes have been analyzed. The symmetry nature of the vectors fields are visualized with Stream Density Plot by using Mathematica.

Acknowledgements

I would like to thank Dr Khin Khin Win, Professor and Head, Department of Physics, University of Yangon, for her kind permission and encouragements to carry out this work.

I would like to express my gratitude to all of the Professors, Department of Physics, University of Yangon, for their kind permission and encouragemen ts to undertake this research paper.

Special thanks are due to Professor, Dr Thant Zin Naing, Retired Pro-Rector (Admin), International Theravāda Buddhist Missionary University, for his valuable guidance and helpful advice to carry out this work.

References

Adler.R, Bazin. M, Schiffer. M, (1975). *Introduction to General Relativity*. New York: Mc.Graw-Hill.

Carroll, S.M., (2004). *Introduction to General Relativity Spacetime and Geometry*, San Francisco, Addison-Wesley.

Hawking S.W, Ellis G.F.R, (1973). *The large scale structure of space-time*, Cambridge: Cambridge Univ. Press.

Mc Mahon. D, (2006). *Relativity Demystified*, New York: McGraw-Hill.

Misner.C.W, Throne. K.S, Wheeler. J.A, (1973). *Gravitation*, San Francisco: W.H. Freeman and Company.

Stephani.H, (2004). *Relativity: An introduction to Special and General Relativity*, Cambridge: Cambridge University Press.

Schutz. B.F, (2009). *A First Course in General Relativity*, Cambridge: Cambridge University Press.

Weinberg.S, (1972). *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity,* Hoboken:J. Wiley & Sons.